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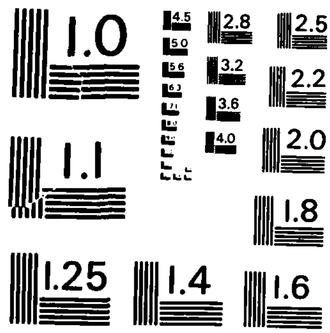

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THE CHINESE REMAINDER PROBLEM AND POLYNOMIAL INTERPOLATION

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ABSTRACT

Let

$$m_i (i=1,\dots,n) \tag{1}$$

be positive integers pairwise relatively prime. The Chinese Remainder Problem is to find a solution x of the n congruences

$$x \equiv a_i (\mathrm{mod}\ m_i) \quad (i=1,\dots,n) \quad . \tag{2}$$

where the integers a_i are given. From Marcel Riesz I learnt orally that this problem is an analogue of the problem of finding a polynomial $P(x)$ of degree $n-1$ which solves the interpolation problem

$$P(x_i) = y_i (i=1,\dots,n) \ (y_i \text{ given and also distinct } x_i) \quad . \tag{3}$$

This is solved by Lagrange's interpolation formula

$$P(x) = \sum_{i=1}^{n} y_i L_i(x) \tag{4}$$

where $L_i(x)$ are the fundamental functions satisfying

$$L_i(x_j) = \delta_{ij} \quad . \tag{5}$$

Also (2) can be similarly solved by determining the $b_i (i=1,\dots,n)$ satisfying the congruences

$$b_i \equiv \delta_{ij} (\mathrm{mod}\ m_j) \tag{6}$$

Theorem 1. *A solution of the system (2) is given by*

$$x = \sum_{i=1}^{n} a_i b_i \quad . \tag{7}$$

Besides recording this analogy of Marcel Riesz, the author's contribution is the following remark: Just as Newton solves the problem (3) successively with his formula using successive divided differences, it is convenient to solve the system (2) successively obtaining

Theorem 2. *The integer*

$$x = a_1 + d_1 m_1 + d_2 m_1 m_2 + \dots + d_{n-1} m_1 m_2 \cdots m_{n-1} \tag{8}$$

is a solution of (2) if we determine the $d_i (i=1,\dots,n-1)$ *successively by the congruences*

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$$
(9)\qquad
\begin{aligned}
& a_1 + d_1 m_1 \equiv a_2 \pmod{m_2} \\
& a_1 + d_1 m_1 + d_2 m_1 m_2 \equiv a_3 \pmod{m_3} \\
& \quad \vdots \\
& a_1 + d_1 m_1 + \cdots + d_{n-1} m_1 \cdots m_{n-1} \equiv a_n \pmod{m_n} \;.
\end{aligned}
$$

Indeed, from (9) we find that

$$x = a_1 + d_1 m_1 + \cdots + d_{k-1} m_1 m_2 \cdots m_{k-1} \equiv a_k \pmod{m_k}$$

for $k = 1,\ldots,n$.

AMS (MOS) Subject Classifications: 10A10, 41A05

Key Words: Chinese Remainder Problem, Polynomial Interpolation

Work Unit Number 6 - Miscellaneous Topics

SIGNIFICANCE AND EXPLANATION

The Chinese Remainder Problem (Ch.R.P) is to find an integer x such that

$$x \equiv a_i (\text{mod } m_i) \quad (i=1,\ldots,n) \quad ,$$

where m_i are pairwise relatively prime moduli and a_i are given integers. In the 1950's I learnt orally from Marcel Riesz that the CH.R.P. is an analogue of the polynomial interpolation problem

$$P(x_i) = y_i (i=1,\ldots,n) \quad , \quad P(x) \in \pi_{n-1} \quad ,$$

and that the Ch.R.P. can be solved by an analogue of Lagrange's interpolation formula. The author now adds the remark that the Ch.R.P. can be solved, even more economically, by an analogue of Newton formula using successive divided differences.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE CHINESE REMAINDER PROBLEM AND POLYNOMIAL INTERPOLATION

I. J. Schoenberg

Let

(1) $m_i (i=1,\ldots,n)$ be positive integer s.t. $(m_i, m_j) = 1$ if $i \neq j$.

The Chinese remainder problem is as follows

<u>The Problem</u>. <u>Given the integers</u> $a_i (i=1,\ldots,n)$ <u>we are to find an interger</u> x <u>satisfying the congruences</u>

(2) $$x \equiv a_i (\text{mod } m_i) \quad , \quad (i=1,\ldots,n) \quad .$$

Sometime in the nineteen-fifties Marcel Riesz visited the Universit; c Pennsylvania and told us informally that the problem (2) can be thought o1 is an analogue of the problem of finding a polynomial $P(x)$ of degree $n-1$ solving the interpolation problem

(3) $$P(x_i) = y_i (i=1,\ldots,n), \quad (y_i \text{ given and also distinct } x_i) \quad .$$

This problem is solved by Lagrange's formula

(4) $$P(x) = \sum_1^n y_i L_i(x) \quad ,$$

where the fundamental functions $L_i(x)$ are defined by

(5) $$L_i(x_j) = \delta_{ij}, \quad (i,j=1,\ldots,n) \quad .$$

Similarly, if we define the integers b_i by the congruences

(6) $$b_i \equiv \delta_{ij} (\text{mod } m_j) \quad (i,j=1,\ldots,n) \quad ,$$

we have

<u>Theorem</u> 1. <u>A solution of the system (2) is given by</u>

(7) $$x = \sum_1^n a_i b_i \quad .$$

Indeed, as soon as we have the b_i satisfying (6), we easily see that the integer x satisfies (2). Clearly the integers a_i are the analogues of

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the y_i of (3), while the integers b_i of (6) are the analogues of the fundamental functions $L_i(x)$ of (5).

Our solution of (2) by means of (6) is essentially also the solution as given in [1, 66-71] and [2, 49-51] without mentioning the analogy with Lagrange's formula (4).

Besides recording Riesz's remark, the author's contribution is the following remark: Newton solves the interpolation prolem (3) successively using successive divided differences. Applying Newton's idea to the solution of the congruences (2) we obtain the following procedure:

Determine the integers

$$d_i (i = 1,2,\dots,n-1) \tag{8}$$

so as to satisfy the $n-1$ congruences

$$\begin{aligned} & a_1 + d_1m_1 \equiv a_2 \pmod{m_2} \\ & a_1 + d_1m_1 + d_2m_1m_2 \equiv a_3 \pmod{m_3} \\ & \quad \vdots \\ & a_1 + d_1m_1 + d_2m_1m_2+\dots+d_{n-1}m_1m_2\cdots m_{n-1} \equiv a_n \pmod{m_n} \ . \end{aligned} \tag{9}$$

Notice he triangular shape of this system: We determine first a value of m_1, then m_2 a.s.f. The d_i having been determined we have

Theorem 2. A solution of the system (2) is given by

$$x = a_1 + d_1m_1 + d_2m_1m_2+\dots+d_{n-1}m_1m_2\cdots m_{n-1} \ . \tag{10}$$

Indeed, from (9) we find that

$$x \equiv a_1 + d_1m_1+\dots+d_{k-1}m_1m_2\cdots m_{k-1} \equiv a_k \pmod{m_k}$$

for $k = 1,2,\dots,n$, because of the $(k-1)$st congruence (9).

Remarks. 1. The seond Newton approach is slightly more economical: While the Lagrange approach required to find the n integers b_i, the Newton approach required to determine only $n-1$ integers $d_i (i=1,2,\dots,n-1)$.

2. The analogy with Newton's solution of (3): The d_i of (10) correspond to the successive divided differences, and the m_i are the analogues of the $x-x_i$.

REFERENCES

1. G. E. Andrews, Number Theory, W. B. Saunders Co., Philadelphia, 1971.
2. Emil Grosswald, Topics from the Theory of Numbers, The Macmillan Co., New York, 1966.

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ABSTRACT (continued)

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(8) $$x = a_1 + d_1 m_1 + d_2 m_1 m_2 + \dots + d_{n-1} m_1 m_2 \cdots m_{n-1}$$

is a solution of (2) if we determine the $d_i(i=1,\dots,n-1)$ successively by the congruences

(9) $$\begin{aligned} &a_1 + d_1 m_1 \equiv a_2 (\bmod\ m_2) \\ &a_1 + d_1 m_1 + d_2 m_1 m_2 \equiv a_3 (\bmod\ m_3) \\ &\quad\vdots \\ &a_1 + d_1 m_1 + \dots + d_{n-1} m_1 \cdots m_{n-1} \equiv a_n (\bmod\ m_n) \quad . \end{aligned}$$

Indeed, from (9) we find that

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for $k = 1,\dots,n$.

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